8. EUCLIDEAN GEOMETRY

§8.1. Euclid's Academy

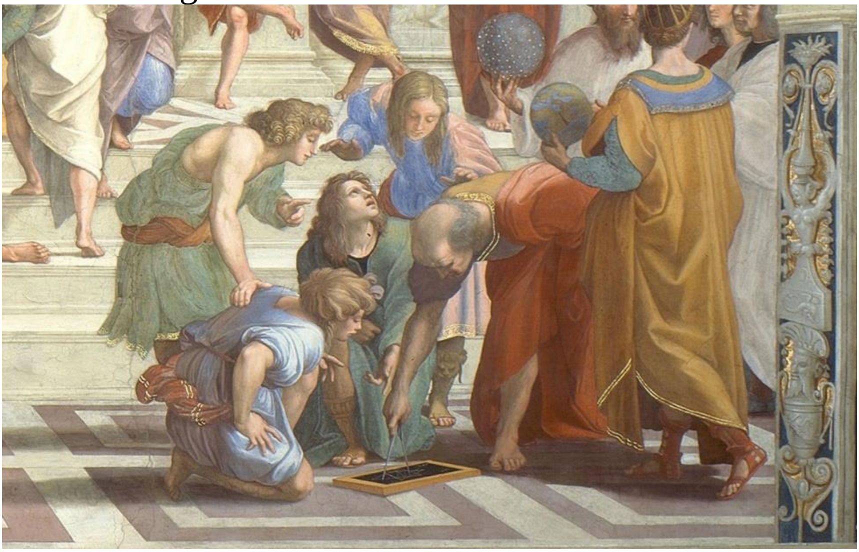
Mathematics can't claim to be the world's oldest profession. But as an intellectual activity it is certainly one of the oldest. Of course mathematics only became a profession around the time of the Renaissance. But historians believe that mathematics has been practised for many thousands of years.

The motivation was practical. It was to serve the needs of commerce. There were only two branches of mathematical knowledge back then: arithmetic and geometry.

Arithmetic was developed in order to support book-keeping (though records of financial transactions were written on stone, or papyrus sheets, not in books). Geometry was developed as an aid to surveying. The word 'geometry' comes from the Greek, meaning 'measuring the earth'.

Euclid, in the 4^{th} century BC, was a Greek who is credited with making a systematic intellectual discipline out of the many rules of thumb that were previously in use. This was at a time long before universities and it is believed by historians that Euclid surrounded himself by disciples, probably much younger than himself. He ran something between an academy and a research school.

One imagines them sitting around a sandy square in Athens, drawing diagrams in the sand and debating geometric ideas. They may have used the Socratic method where dialogue and discussion was used to locate truths.



It has been said that, before Euclid, geometry employed the scientific method. It was suggested that Pythagoras formulated his famous theorem by examining a large number of right-angled triangles. That's how we might do it today, but papyrus was scarce back then. It was long known that the 3-4-5 triangle was right-angled and somebody might have stumbled on the 5-12-13 example. Then perhaps somebody else noted the arithmetic pattern in these numbers, but a proof was still to come.

No doubt the discussion that took place between the Euclideans involved a lot of argument along the lines of 'surely …' or 'it would seem reasonable that …'. But I imagine that short arguments would have been put forward that provided logical bridges between some of these geometrical statements. "Well we all know that … and so it follows that … (perhaps with a few extra construction lines) …".

We have no way of knowing what went on in these discussions, but I can imagine Euclid himself coming up with the idea of systematising all these bridges and creating a unified structure that built geometry from a small number of postulates, or axioms. These were very basic statements which could be accepted intuitively. For example, "given any two distinct points there exists exactly one straight line passing through them". Perhaps this would have been backed up by a small amount of experimentation, but I'm sure you've seen enough examples to know in your heart that it's true. Of course you probably never considered the possibility that there might be many straight lines joining them that were so close to each other that your eye couldn't tell the difference.

Euclid's *magnum opus* is his *Elements*. This has been a standard text-book in universities and schools throughout many centuries. It was used, both in the original Greek, and later in translation, up until the end of the 19th century. It is said that it is second only to the Bible in the number of editions (over a thousand) printed since the first printed edition in 1482.

§8.2. Euclid's Formulation of Geometry

Euclid had the vision of formulating geometry in such a way that the truth of the theorems didn't rest on the intuition of the individual. By setting down axioms, and building everything logically from these axioms, everyone who accepted the axioms would have to accept all the theorems. And these axioms were considered to be self-evident.

Euclid's formulation consists of five sections.

(1) 23 Definitions

By rights, some of these should be undefined entities, but Euclid feels the need to define even these. So he defines a point as that of which there is no part and a line as a length without breadth. The first is very vague and the second is meaningless without first having defined length and breadth. But it does reveal the fact that, for Euclid, lines are finite. We would call them **line segments**. However he includes a postulate (axiom) to the effect that any line can be extended where necessary.

He defines a circle as a 'plane figure contained by a single line (called a **circumference**), such that all of the straight lines radiating towards the circumference from one point amongst those lying inside the circle are equal to one another'. But he doesn't actually define radius, or even length. The underlying number system is unstated.

Angles are defined as 'the inclination of lines to one another'. This is a case of defining one thing in terms of a synonym. Not very useful.

(2) 5 Postulates

These are what we would call his axioms. The first three are actually constructions. He doesn't exactly say that what is constructed is unique, but this is implied.

(E1) Through any pair of points there a (unique) line.

(E2) Any (finite) line can be produced.

By considering lines as infinite we can avoid the need for this postulate.

(E3) There exists a (unique) circle with any point as centre and with any radius.

A better version would be to say that, given any two distinct points there is exactly one circle whose centre is the first point and which passes through the second. This avoids the need for defining radius.

In his definition Euclid seems to be considering the circle to include the interior, but when he starts intersecting a circle with a line it's clear that he means the circumference.

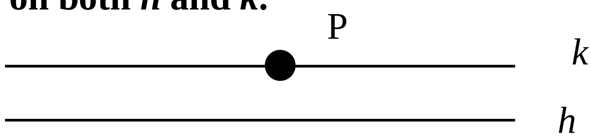
(E4) All right-angles are equal.

Euclid defines a right-angle by saying that when a line meets another and the adjacent angles are equal then they are right angles. This postulate doesn't seem to add anything.

(E5) If a line cuts two other lines and the internal angles total less than two right angles, then the lines are not parallel.

Postulate (E5) was reformulated by Playfair (1748-1819) as follows:

(P5) Given a line *h*, and a point P that does not lie on *h*, there is a unique line *k* such that P lies on *k* and no point lies on both *h* and *k*.



(3) 5 Common Notions

These have more to do with the underlying number system than to geometry such as 'things that are equal to the same thing are equal to one another'.

When I began this chapter, I thought that I could incorporate Euclidean Geometry by simply defining the

Euclidean Plane to be $\mathbb{R}^2$. Points are just vectors in $\mathbb{R}^2$. Lines, triangles and circles are just certain sets of points in $\mathbb{R}^2$. So, all I'd have to do would be to prove the Euclidean axioms and then everything would follow as in Euclid's *Elements*. Euclidean Geometry would, as everything else, sit firmly on the foundation of the ZF axioms for set theory.

But I began to realise that, as good a job as Euclid did (he was far ahead of his time), he didn't quite achieve his goal of making plane geometry stand alone on his axioms, without the need for geometrical intuition.

Essentially he had the axioms for an affine plane, a geometry without measurement. By just taking Axioms (E1) and (P5) we have:

(A1) Every pair of distinct points lies on exactly one line.
(A2) Given a line, *h*, and a point P that doesn't lie on *h*, there is exactly one line, *k*, such that P lies on k and no point lies on both *h* and *k*.

Euclid includes notions of length and angle, but is vague as to what they are. He has the notion of **equal** line segments, which appears to have an underlying concept of similarity. Although he doesn't say so explicitly he seems to consider two intervals to be equal if one can be obtained from the other by a rotation followed by a translation. This is clearly his intention with his definition of a circle with equal radii. But in a few places he needs to add two intervals if we have three collinear points.

We could tell our disembodied angel that ‘same length’ is an equivalence relation and include an axiom that if B lies between A and C on a line then AB + BC = AC.

Angles are even more of a problem. We don’t just want to have equal angles, but we frequently need to add angles.

§8.3. The Euclidean Plane

So I had to abandon any idea of teaching my disembodied angel Euclidean Geometry by following Euclid’s *Elements*. In any case I don’t want her to have to accept any more axioms than those of set theory. So I would proceed a follows.

Having developed $\mathbb{R}$, I introduce $\mathbb{R}^n$ and so define points and lines. The line joining **u** and **v** would be the set $\{(1 - \lambda)\mathbf{u} + \lambda\mathbf{v} | \lambda \in \mathbb{R}\}$. This automatically gives a direction to the line, should we need it and enables us to define $\mathbf{w} = (1 - \lambda)\mathbf{u} + \lambda\mathbf{v}$ to lie between **u** and **v** if

$$0 < \lambda < 1.$$

I would then define the dot product in the usual way and so define **distance** and **lengths**. **Angles** can be defined by:

$$\angle \mathbf{ABC} = \cos^{-1}\frac{(\mathbf{A} - \mathbf{B}).(\mathbf{C} - \mathbf{B})}{|\mathbf{A} - \mathbf{B}|.|\mathbf{C} - \mathbf{B}|}.$$

After playing with *n*-dimensional Euclidean Geometry I would focus on the Euclidean Plane $\mathbb{R}^2$. I would identify it with the field of complex numbers, and show that every non-zero point (complex number) can be expressed as $r(\cos\theta + i \sin\theta)$ for $r > 0$.

I would define angles afresh by:

$$\angle ABC = \arg(C - B) - \arg(A - B).$$

Angles would be considered as real numbers modulo 2π.

And, if the angel insisted, I would attempt to reconcile this with the previous definition. (I leave this as an exercise, which means I haven't bothered to do it myself.) Clearly addition of angles works properly in that $\angle AOB + \angle BOC = \angle AOC$.

Things become somewhat difficult when it comes to areas.

§8.4. Defining Areas

How do we define area? We first learnt that the area of a rectangle is 'length times breadth'. No difficulty in that. And which side of the rectangle is the length? Clearly it doesn't matter because lengths are real numbers and real numbers commute under multiplication.

We probably next saw a rectangle cut in two by a diagonal and, seeing that the resulting right-angled triangles are congruent we decided that it was obvious that the area of a right-angled triangle is half the base times the perpendicular height.

Now a triangle has three possible bases, each a corresponding perpendicular height. I don't know whether it ever occurred to you that you might get three different areas, depending which you took as the base. I thought not. It's intuitively obvious that it makes no difference.

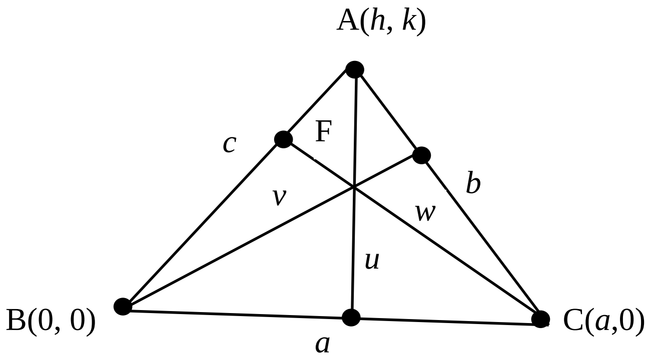
Is the area of a triangle 'well-defined'? We somehow have the concept of area hard-wired into our brains. Perhaps we think of area being related to the amount of paint we'd need to paint it. It's obvious that it doesn't matter which way up we stand it.

But if we are to do things properly, and certainly if we want the disembodied angel to be convinced, we have to prove well-definedness.

Theorem 1: The area of a triangle as 'half the base times the perpendicular height' is independent of which side is taken as the base.

Proof: Let ABC be a triangle, with $|BC| = a$, $|CA| = b$ and $|AB| = c$. Let u, v, w be the perpendicular distances of A, B, C from the opposite sides be u, v and w respectively.

For convenience, locate B at the origin, C at $(a, 0)$ and A at (h, k).



The perpendicular distance of the point (x_1, y_1) from the line $px + qy + r$ is $\left|\dfrac{px_1 + qy_1 + r}{\sqrt{p^2 + q^2}}\right|$.

The equation of AB is $y = \dfrac{k}{h}x$, that is $kx - hy = 0$.

Hence $u = \dfrac{ka}{\sqrt{k^2 + h^2}}$. Now $c = \sqrt{k^2 + h^2}$ so $cu = ka$.

The equation of AC is $\dfrac{y}{x - a} = \dfrac{k}{h - a}$, that is

$kx + (a - h)y - ka = 0$.

Hence $v = \dfrac{ka}{\sqrt{k^2 + (a - h)^2}}$. Now $b = \sqrt{k^2 + (a - h)^2}$, so

$bv = ka$. And $u = k$, so $au = ka$.

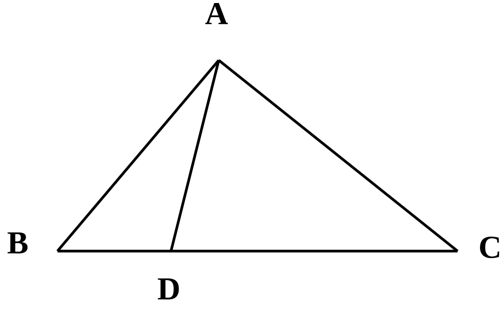
So the area of the triangle is $\frac{1}{2}ka$ no matter which side is taken as the base.

In primary school we learnt that the area of a polygon can be found by triangulating it – divide it up into triangles. We could take this as the definition of the area of a polygon. The trouble is the question of well-definedness.

You and I might divide a polygon into triangles in two completely different ways. How do we know that we always get the same answer. It's seems obvious that we do get the same answer but I'd rather not have to include this among our axioms.

Suppose we triangulate a triangle. Why would you want to do that? Never mind. If area is additive then cutting a triangle into smaller triangles shouldn't affect the total area. Let's just consider a triangulation into two triangles.

Theorem 2: Let ABC be a triangle and D lie on BC between B and C. Then |ΔABC| = |ΔABD| + |ΔADC|.



Proof: By Theorem 1 we are free to choose BC, BD and DC, respectively, as the bases. Clearly all three triangles have the same perpendicular height. Let it be h.

Then $|\Delta ABC| = \tfrac{1}{2}h|BC| = \tfrac{1}{2}h(|BD| + |BC|)$

$$= \tfrac{1}{2}h|BD| + \tfrac{1}{2}h|BC|$$

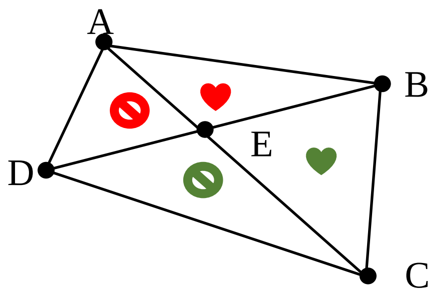
$$= |\Delta ABD| + |\Delta ADC|.$$

Let's take a quadrilateral. There are infinitely ways of triangulating a quadrilateral, but if we want to minimise the number of triangles there are just two possibilities. We divide the quadrilateral into two triangles, using one of the two diagonals. Yes, we can prove that we get the same area either way.

Theorem 2: Let $\mathcal{P}$(ABCD) be a quadrilateral. Then

$$|\Delta ACD| + |\Delta ABC| = |\Delta ABD| + |\Delta BCD|.$$

Proof: Let E = AC $\cap$ BD.



Then $|\Delta ACD| = |\Delta AED| + |\Delta DEC|$ by Theorem 2.

$|\Delta ABC| = |\Delta ABE| + |\Delta BCE|$, again by Theorem 2.

Hence $|\Delta ACD| + |\Delta ABC| = (|\Delta AED| + |\Delta DEC|) + (|\Delta ABE| + |\Delta BCE|)$

$= (|\Delta AED| + |\Delta ABE|) + (|\Delta ECD| + |\Delta BCE|)$

$= |\Delta ABD| + |\Delta BCD|$.

As this example suggests, given two triangulations we can refine each of them to a common triangulation. Here's a clue as to how we can prove that the area of a polygon is independent of the triangulation. All we need is a proof that if a triangle is triangulated, its area is the sum of the areas in the triangulation. Possibly this could be done by induction in some way.

But here's where I grind to a halt. I haven't been able to prove this. Nor have I been able to find a proof in the literature. In fact I cannot find any reference to the additivity of area being an issue.

If you think I'm making a mountain out of a molehill, you'd be right. Of course area is additive. But if you want to really understand something you can't just say, "of course".

Euclid was very careful in building up geometry in a rigorous and systematic way, but he fell down when it comes to area. In fact he never defines area and nor does he take it as an undefined concept subject to his axioms.

The additivity of area seems to be hard-wired into our brains, like the additivity of volume. Hard-wired it might be, but we don't seem to be born with it. The psychologist Jean Piaget (1896-1980) carried out a famous experiment, repeated by countless other psychologists since, where children are shown two identical glasses of water. When they see one glass poured into a taller but narrower glass they are convinced, up to the age of about 8 or 9, that the taller glass now contains more water than the other one. A similar experiment has been carried out with areas and the results were much the same.

Of course there's more to area than just areas of polygons. For regions involving curved boundaries we need calculus. For more general regions we need a branch of mathematics called Measure Theory.

Anyway, this is as far as I'm going to take Euclidean Geometry in these notes. If you want a development of the subject beyond this introductory stage, please consult my notes in *Geometry vol 1*. We will now further develop Set Theory itself.

I will develop the theory of infinite numbers. For a start there are two types of number – cardinal numbers and ordinal numbers.

You may have met the distinction between cardinal and ordinal finite numbers. The distinction there is purely linguistic. Cardinal numbers count the number of elements in a set. Ordinal numbers describe the position of an element when a set is ordered. So 'five' is a cardinal number, while 'fifth' is the corresponding ordinal number. With infinite numbers the distinction is far from just linguistic. A single infinite cardinal number corresponds to infinitely many different infinite ordinal numbers.

Yes, I'm using the plural when it comes to talking about infinite numbers. We'll use the notation **#S** to denote the size of the set S. Forget the symbol ∞ for infinity. There are, in fact, infinitely many infinite numbers (both cardinal and ordinal).

The smallest infinite cardinal number is #ℕ, the size of the set {0, 1, 2, 3, …}. When Georg Cantor developed the theory of infinite cardinal numbers in the 19th century, he used the symbol $\aleph_0$ to denote # ℕ.

There are sets that appear to be much bigger than ℕ, such as ℤ (integers) and ℚ (rational numbers), but

$$\#\mathbb{Z} = \#\mathbb{Q} = \aleph_0.$$

The jump comes when we pass from the rational numbers to the real numbers. I shall denote #ℝ by $\aleph_1$.

In fact $\#\mathbb{R} = \#\,\mathbb{C} = \aleph_1$.

Now many books use the symbol $\aleph_1$ for 'the next cardinal number after $\aleph_0$'. What's the difference?

Most books define $\aleph_1$ to be the next cardinal number after $\aleph_0$. The problem is that we don't know what that number is.

Using my notation we can ask the question, "is there an infinite cardinal number between $\aleph_0$ and $\aleph_1$?" The interesting fact is that we cannot answer that question. It's not that no mathematician has ever been clever enough to come up with an answer. No mathematician ever will!

The reason we can be so certain is that there's a proof that no proof can possibly exist for the statement, "there is no cardinal number between $\aleph_0$ and $\aleph_1$". There's another proof that no proof can possibly exist for the statement, "there exists a cardinal number between $\aleph_0$ and $\aleph_1$". In other words the answer to the question is unknowable.

The statement that there is no infinite number between $\aleph_0$ and $\aleph_1$ is called the **Continuum Hypothesis**. So the Continuum Hypothesis can never be proved true and it can never be proved false. If you choose to believe it, fine. Nobody can ever prove that you're wrong. If you choose to prove it false, fine. Nobody can ever prove that you're wrong.

I take the pragmatic view that if we could ever exhibit a *specific* infinite number between $\aleph_0$ and $\aleph_1$, we would have contradicted the second statement above. That's not to say I can prove that there are none – just that if these exist we can never get our hands on them.

Personally I choose to believe in the Continuum Hypothesis on the grounds that there's no point in believing the existence of phantom numbers that one can never do anything with. Assuming the Continuum Hypothesis the next number after $\aleph_0$ is the cardinal number $\mathbb{R}$, which I call $\aleph_1$.

So hang onto your hats. You're in for a wild ride! You'll be learning about some mathematics which is, on the one hand, perfectly sound, but on the other it is perfectly weird. I call it *Mathematics at the Edge of the Rational Universe*. In fact I have some notes that go by that very name. These are written for the mathematical layman. These present notes are written at a more sophisticated level, as is appropriate for a mathematics student in his or her third year.